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Appearance of a Relativistically Rotating Disk

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The appearance of a rotating disk, as perceived by a corotating observer in accordance with two operational procedures is discussed, and the results compared. It is noted that naive generalizations of operational procedures which correctly represent the disk geometry when stationary lead to mutually contradictory pictures when the disk rotates.

1. INTRODUCTION

The rotating disk is probably the simplest of all noninertial frames of reference which illustrates, in the context of general relativity, the existence of non-Euclidean spatial geometry, and the distinction between space-time null geodesics, representing photon trajectories, and the spatial geodesics of the underlying reference frame. The disk has, therefore, received considerable historical (Silberstein 1921, Møller 1952) and contemporary (Grøn 1975, Brown 1977, McFarlane and McGill 1978) attention. Equally a great deal of interest (Penrose 1959, McGill 1968, Scott and Van Driel 1970) has been centered on the "appearance" of bodies in rapid, uniform, and rectilinear motion as perceived by an inertial observer. It is perhaps surprising, therefore, that, whereas some work (Grøn 1975) has been done on the appearance of the disk as seen by inertial observers, nevertheless no substantial analysis of the disk's appearance as "seen" by a corotating observer has been undertaken. It is to this last task that this paper is addressed.

The idea of "appearance" of the rotating disk as seen by a corotating observer requires elucidation: Let an operational procedure be given which, by a combination of measurement and calculation, assigns to every point m of the disk some distance d from the corotating observer and an angle φ

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relative to a given base line at the observer, and let these be represented in the Euclidean plane by images whose plane polar coordinates relative to the point representing the observer's location are (d, φ); then we shall interpret the "appearance" of the disk as seen by the observer in accordance with the given operational procedure as the appearance of the disk's image under the map $m \mapsto (d, \varphi)$.

This interpretation is in accord with the various "appearances" as defined in special relativity, where the radar, camera, stereographic, and other impressions of a moving body may all be understood in terms of an appropriate operational procedure and an identification of the image of the body with points of a two- or three-dimensional Euclidean image space. There is, however, an essential difference between the special relativistic appearance problems, where the distortion of the image is due entirely to the finite transmission time for light from a distant object to reach the observer, and the problem of the rotating disk, where an additional distortion always exists associated with the mapping of the disk's non-Euclidean spatial geometry into the Euclidean plane.

We shall in this paper consider two operational procedures defining the appearance of the disk; the first induces only a "geometrical" distortion associated with the representation of a curved in a flat space, whereas the second, using light to define the location of distant points, is dominated by first-order "time-of-flight" effects.

2. PRELIMINARIES

All the basic results listed in this section are developed at greater length in McFarlane and McGill (1978), to which the reader is referred for details.

The rotating disk may be minimally regarded as a system of particles of negligible mass rotating relative to an underlying inertial frame at constant angular velocity ω about a fixed axis normal to the disk plane. In terms of inertial polar coordinates $(\bar{r}, \bar{\theta})$ in the plane of the disk, and the inertial proper time $\bar{t}$, we may elect as characterizing the rotating disk the event coordinates (r, θ, t) related to their inertial counterparts by

$$\bar{r}=r, \qquad \bar{\theta}=\theta+\omega t, \qquad \bar{t}=t, \qquad \omega^2 r^2/c^2<1 \tag{1}$$

each fixed pair of coordinates (r, θ) identifying a unique point on the rotating surface.

In terms of the unbarred coordinates the metric of flat space-time takes the form

$$ds^2=dt^2(1-\omega^2 r^2/c^2)-c^{-2}(dr^2+r^2 d\theta^2+2\omega r^2 d\theta dt) \tag{2}$$

and results in a non-Euclidean spatial geometry for the disk, determined by the line element

$$dl^2 = dr^2 + r^2(1 - \omega^2 r^2/c^2)^{-1} d\theta^2 \tag{3}$$

Solution of the space-time null geodesic equations results in the following equations for the paths of photons:

$$r\cos(\theta - \theta_0 + \omega t) = r_0, \qquad r\sin(\theta - \theta_0 + \omega t) = \sigma ct \tag{4}$$

in which $(r_0, \theta_0, 0)$ are the event coordinates of the closest approach of a photon to the disk center, and in which $\sigma = \pm 1$ is a parameter which defines the sense of propagation of the photon on its trajectory, the velocity of light c being always hereinafter assumed positive. There exists also a family of singular solutions of the form

$$r^2 = c^2 t^2, \qquad \theta - \theta_0 + \omega t = 0 \tag{5}$$

corresponding to photon tracks passing through the origin of coordinates, and making an inertial polar angle $\bar{\theta} = \theta_0$ with the base line $\bar{\theta} = 0$.

A mathematically similar integration yields the general forms

$$r\cos(\theta - \theta_0 + \omega^2 y_0 r_0 l/c^2) = r_0, \qquad r\sin(\theta - \theta_0 + \omega^2 y_0 r_0 l/c^2) = y_0 l \tag{6}$$

for the spatial geodesics, in which (r_0, θ_0) is the point of closest approach of the geodetic arc to the origin, l the (signed) distance along the geodesic connecting (r_0, θ_0) to (r, θ), and y_0 the auxiliary parameter $(1 - \omega^2 r_0^2/c^2)^{-1/2}$. Singular solutions passing through the origin also exist, and are described by the system

$$r^2 = l^2, \qquad \theta = \theta_0 \tag{7}$$

This last observation enables a simple physical interpretation to be given to the coordinates (r, θ): r is the geodetic distance of the point from the disk center, and θ is the angle at the origin between the geodesic from (r, θ) and the geodesic from $(r, 0)$. It will scarcely be an abuse of language, therefore, to refer to the coordinates (r, θ) as disk polar coordinates.

3. THE "GEOMETRICAL" APPEARANCE OF THE ROTATING DISK

We now turn our attention to the appearance of the rotating disk, and consider initially the world view of a corotating observer at (ρ, φ) whose (mistaken) belief in the Euclidean spatial geometry of the disk is unshakable. So as to represent the "true" form of the disk, such an observer first performs the following sequence of measurements:

[1] He measures his geodetic distance from each distant point (r, θ) and obtains a value d.

[2] He measures, say, the local angle χ at (ρ, φ) between the connecting geodesic to (r, θ) and a reference direction formed by the geodesic to $(c/\omega, \varphi)$, which we may term the outward-facing normal direction.

The observer then constructs a two-dimensional Euclidean representation S of the disk based upon the following procedures:

[3] He represents his own position on the disk by the point in S whose plane polar coordinates are (ρ, φ).

[4] He represents the distant point (r, θ) of the disk by the image point in S at distance d along the straight line at angle χ relative to the outward-facing normal direction from the point (ρ, φ) of S.

The procedure is as illustrated in Figure 1.

We shall now calculate and graphically illustrate the appearance of the disk as "seen" by such an observer, to which end we find it convenient hereinafter to introduce the following dimensionless variables:

$$\xi = \omega\rho/c, \qquad x = \omega r/c, \qquad \lambda = \omega l/c, \qquad \tau = \omega t \tag{8}$$

Our first task will be to determine the spatial geodesic connecting the points (x, θ) and (ξ, φ), these being located on the connecting geodesic by the arc-length parameters λ_1 and λ_2, respectively. The crucial observation is that the equation

$$x_0 y_0(\lambda_2 - \lambda_1) = x\xi \sin[\varphi - \theta + x_0 y_0(\lambda_2 - \lambda_1)] \tag{9}$$

obtained from the system (6) has precisely one real root for the quantity $\eta = x_0 y_0(\lambda_2 - \lambda_1)$, and that this root may be arbitrarily well approximated by the convergent linear iteration

$$\eta_{n+1} = x\xi \sin(\varphi - \theta + \eta_n), \qquad \eta_0 = 0 \tag{10}$$

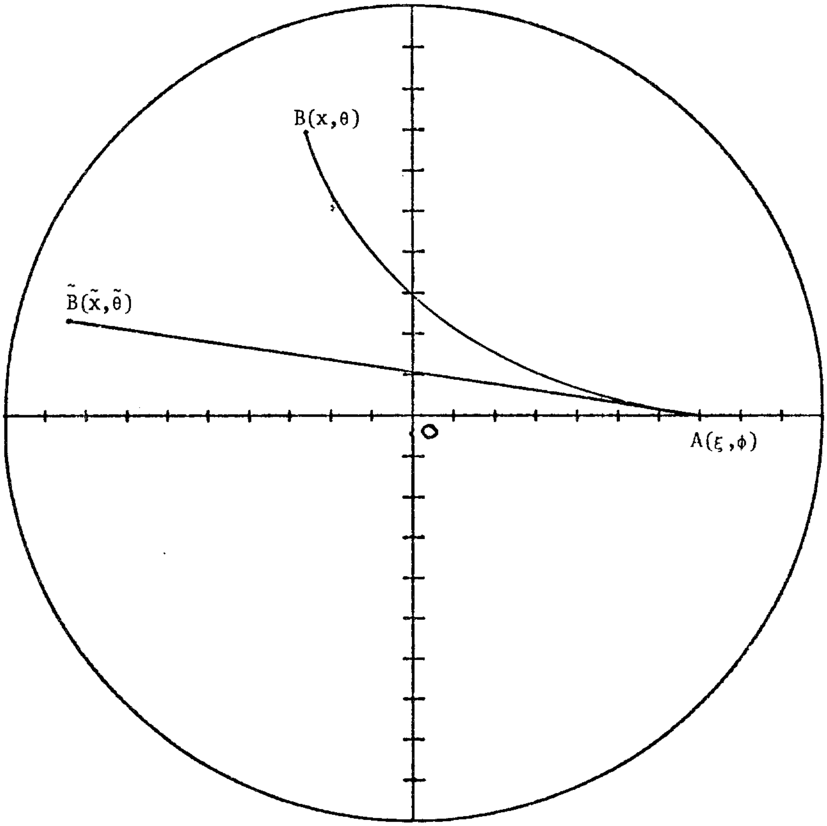


Fig. 1. To illustrate the procedure defining the geometrical appearance of the disk. $A(\xi, \varphi)$ is the point of observation, and $B(x, \theta)$ is the distant point whose geometrical image is $\tilde{B}(\tilde{x}, \tilde{\theta})$.

For the parameter η, when combined with the second auxiliary quantity

$$y_0 x_0^{-1}(\lambda_2 - \lambda_1) = \eta^{-1}\left[x^2 + \xi^2 - 2x\xi\cos(\varphi - \theta + \eta)\right] \tag{11}$$

enables the immediate deduction of x_0 and hence of $\lambda_2 - \lambda_1$, so that the remaining parameters λ_1, λ_2, and θ_0 which describe the geodesic may be obtained by back-substitution into the system (6).

Given these basic parameters we may deduce the (normalized) geodetic distance between (x, θ) and (ξ, φ) as simply

$$d = |\lambda_2 - \lambda_1| \tag{12}$$

and may determine, after some calculation, the implicit equations for the corresponding angle χ,

$$\xi\cos\chi = -\epsilon y_0^2\lambda_2, \qquad \xi\sin\chi = -\epsilon x_0 y_0(1-\xi^2)^{1/2} \tag{13}$$

in which and hereinafter $\epsilon = \text{sign}(\lambda_2 - \lambda_1)$. The above group of equations now facilitates the complete numerical solution of the appearance problem.

It will be convenient for the purpose of diagrammatic illustration to introduce the idea of the standard grid centered upon a point of the rotating disk:

[5] The standard grid centered upon the point (x, θ) comprises a regularly spaces array of contour lines drawn at (normalized) distances $d_\kappa = 0.1\kappa$, $\kappa \in \{1,2,\ldots\}$ from (x,θ), and a regular array of "radial" geodesics starting from (x,θ) and making these angles $\theta_\kappa = \pi\kappa/10$, $\kappa \in \{1,2,\ldots,20\}$ with the local outward normal direction. In the special case of the standard grid centered upon the origin the angles θ_κ are to be interpreted as disk polar angles.

In Figure 2 we have plotted the images, in accordance with the procedures [1]–[4] (the "geometrical" appearance) of the standard grid centered upon the origin as viewed by observers at (Figure 2a) $(\xi,\varphi) = (0.2, 0)$ and (Figure 2b) $(\xi,\varphi) = (0.8, 0)$. Two general features of these graphs may be noted; first, that the disk is "length contracted" in the direction of the inertial motion of (ξ,φ), so that the disk appears oblate, and second that the "radial" geodesics from the disk center are so distorted as to result in an apparent angular magnification of the region nearest to, and a corresponding contraction of the region furthest from, the observer.

That the image contours have the analytic form of concentric ellipses can be seen as follows: In Figure 1 apply the cosine rule to the triangle $0A\tilde{B}$ to deduce, upon substitution from (12) and (13), that the radial coordinate $\tilde{x}$ of $\tilde{B}$ satisfies

$$\tilde{x} = (x^2 - \eta^2)^{1/2} \tag{14}$$

Similarly, on applying the sine rule to the same triangle, we deduce that the corresponding polar angle $\tilde{\theta}$ obeys

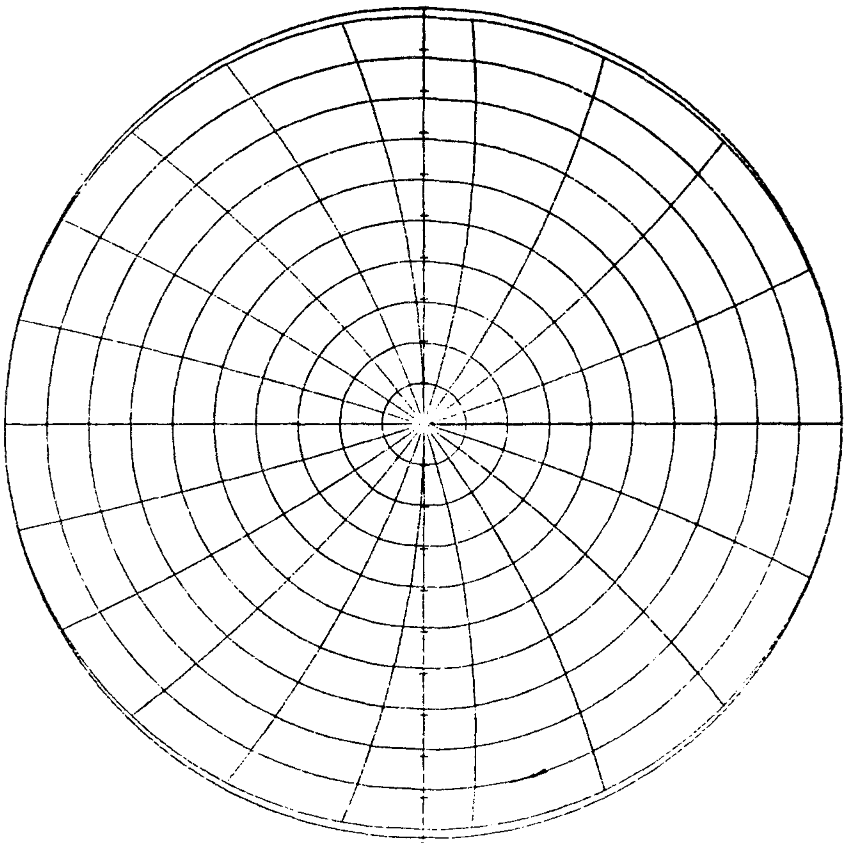
$$\xi\sin\tilde{\theta} = -\epsilon\eta(1-\xi^2)^{1/2}(x^2-\eta^2)^{-1/2} \tag{15}$$

Comparison of these equations with the parametric system

$$\tilde{x}\cos\tilde{\theta} = x\cos\mu, \qquad \tilde{x}\sin\tilde{\theta} = x(1-\xi^2)^{1/2}\sin\mu \tag{16}$$

yields, upon setting $\eta = x\xi \sin \mu$, that the system consisting of (14) and (15) is equivalent to that of (16), and therefore that the contour images are concentric ellipses of semimajor axis x, semiminor axis $x(1-\xi^2)^{1/2}$, eccentricity x, and foci $(x\xi, 0)$, $(x\xi, \pi)$. It is interesting to note that these contours, and indeed the entire representation S of such a naive rotating observer, are precisely those "perceived" by an instantaneously coincident and comoving inertial observer whose image of the disk is as calculated by Grøn (1975).

Let us finally remark that the operational procedure defined above is fully equivalent to an essentially abstract representational map which as-



a

Fig. 2. Geometrical appearance of a standard grid centered on the origin (see text) as perceived by observers at (a) $(\xi, \varphi) = (0.2, 0)$, and (b) $(\xi, \varphi) = (0.8, 0)$.

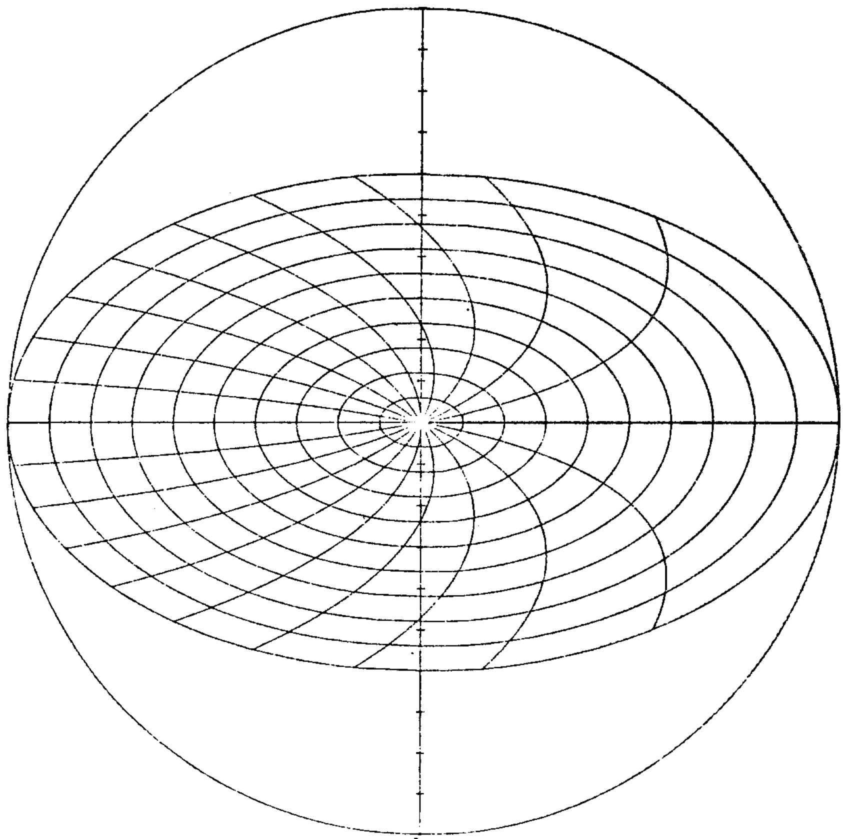


Fig. 2. Continued.

signs points on the disk to points in the image space S, and that in particular from the viewpoint of a central observer the geometrical appearance of the disk is precisely that of a "representational diagram of type (a)" (McFarlane and McGill, 1978), so that the type (a) diagrams of this earlier paper may be regarded as a central observer's view of the disk and certain trajectories thereon. Equally the analysis of this section may be regarded as a elucidation of another type of representational mapping. We shall, however, prefer to emphasize the operational procedure defining this representation so as to contrast it with the "optical" appearance of the disk as defined in the next section.

4. THE "OPTICAL" APPEARANCE OF THE DISK

Let us consider the appearance of the rotating disk as determined by an observer who naively assumes not only that the disk geometry is Euclidean, but also that in the frame of the disk light propagates rectilinearly with speed c. He assigns to each reference point (x, θ) on the disk a corresponding image point obtained in accordance with the following experimental and analytical procedure:

[6] He measures the "round trip" time for a photon transmitted from his position at (ξ, φ) to be reflected from the distant reference point at (x, θ) and to return to its point of origin. The "radar distance" δ of the point (x, θ) from (ξ, φ) is then defined to be $c/2$ times the round trip time, time being measured by a standard clock at (ξ, φ) on the disk.

[7] He measures the local angle, ψ say, made by the incoming photons from (x, θ) to the outward-facing normal at (ξ, φ).

[8] He constructs a two-dimensional Euclidean representation of the disk in which the disk polar coordinates of his own location are identified with the plane polar coordinates of his image point in the representational space S.

[9] Finally, he marks the image of each distant point (x, θ) at distance δ along the straight line from (ξ, φ) at an angle ψ to the local outward normal direction, the resultant point having the plane polar coordinates (x', θ').

The procedure and construction are analogous to that illustrated in Figure 1, but the space-time null geodesic replaces the spatial geodesic connecting (x, θ) to (ξ, φ).

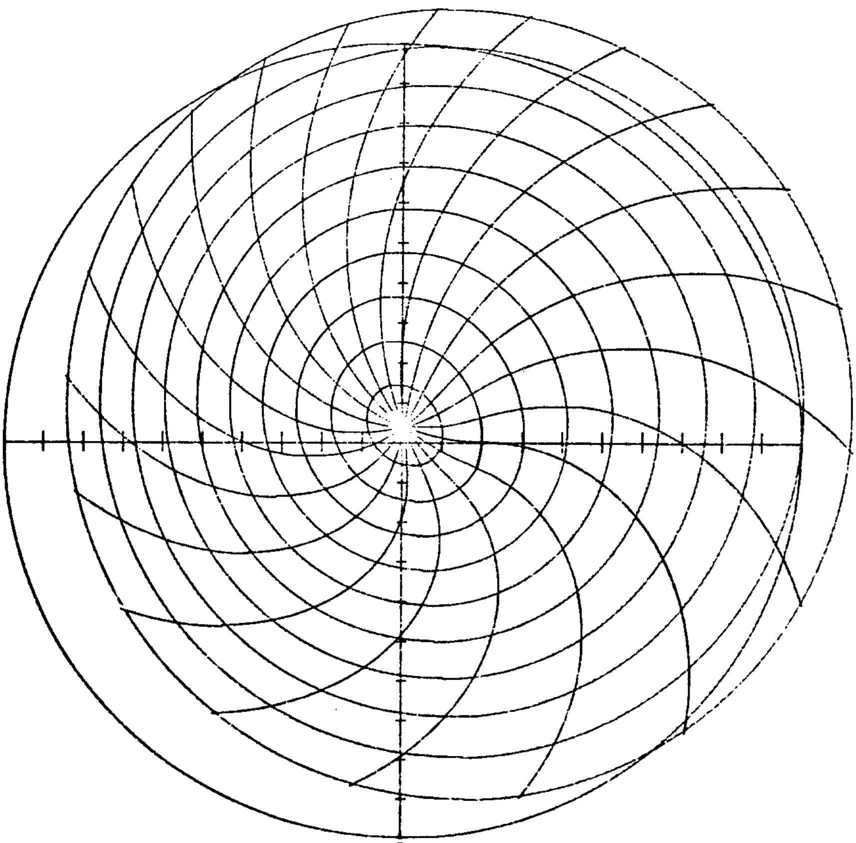
It should be noted that whereas the definition of radar distance is natural and immediate, any purely optical means of defining the apparent angle ψ of a distant point is to some extent artificial, since different angles are made by the incoming and outgoing photons reflected from and transmitted to a distant point. The choice [7], where the observer need only view the reflected light from an object in order to determine its apparent angle, is probably the most natural, has the advantage of computational simplicity when compared with the obvious alternative definition of ψ as the mean of incoming and outgoing photon directions, and yields broadly similar results.

Let us now calculate the "optical" appearance of the rotating disk, as defined in accordance with [6]–[9] above, and begin by determining the space-time null geodesic connecting (x, θ) to (ξ, φ). The starting point is the "time-of-flight" equation

$$(\tau_2-\tau_1)^2=x^2+\xi^2-2x\xi\cos(\varphi-\theta+\tau_2-\tau_1) \tag{17}$$

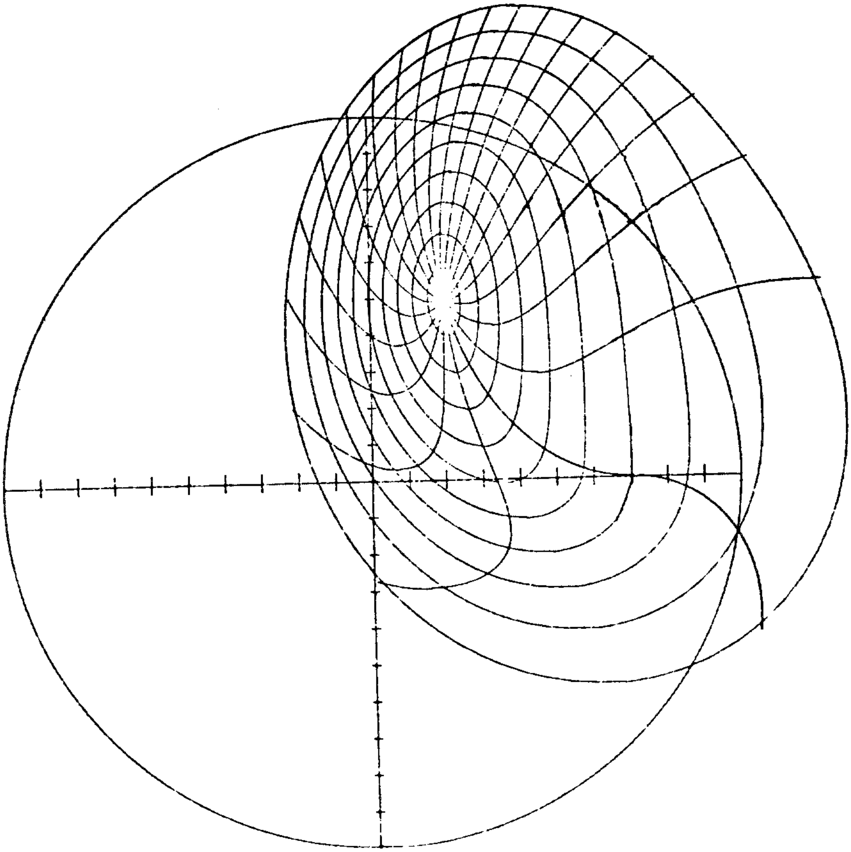
which describes the (normalized) time of flight $\tau_2-\tau_1$ between the transmission event (x, θ, τ_1) of a photon from (x, θ) and the subsequent reception event (ξ, φ, τ_2). This equation has a unique positive solution to which the linear iteration

$$\eta_{n+1}=x^2+\xi^2-2x\xi\cos\left[\varphi-\theta+(\eta_n)^{1/2}\right], \qquad (\eta_n)^{1/2}\mapsto\tau_2-\tau_1 \tag{18}$$



a

Fig. 3. Optical appearance of a standard grid centered on the origin as perceived by observers at (a) (0.2,0), and (b) (0.8,0).



b

Fig. 3. Continued.

is convergent for all starting values sufficiently close to the root. The parameters σ and x_0 then follow directly from the equation

$$\sigma x_0(\tau_2 - \tau_1) = x\xi \sin(\varphi - \theta + \tau_2 - \tau_1), \qquad x_0 > 0 \tag{19}$$

and the quantities τ_1, τ_2, and θ_0 are now deducible from the fundamental system (4). The radar distance δ now follows immediately from (17) and its image equation describing the time of flight between the transmission event (ξ, φ, τ_1) and the reception event (x, θ, τ_2), and the local angle ψ of [7] is given implicitly as

$$\xi\cos\psi = -\tau_2(1-\xi^2)^{1/2}(1-\sigma x_0)^{-1}, \qquad \xi\sin\psi = (\sigma x_0 - \xi^2)(1-\sigma x_0)^{-1} \tag{20}$$

Figure 3 shows the optical appearance, when viewed in accordance with [6]–[9], of a standard grid centered on the origin, diagram 3a representing the appearance from $(\xi, \varphi) = (0.2, 0)$, and diagram 3b from $(0.8, 0)$. The diagrams have been scaled by a factor $(1-\xi^2)^{-1/2}$ about the point (ξ, φ) so as to offset the progressive reduction in their overall size which would otherwise occur with increasing ξ. This is equivalent to choosing inertial proper time in the definition of "radar" distance. Note that the figure may

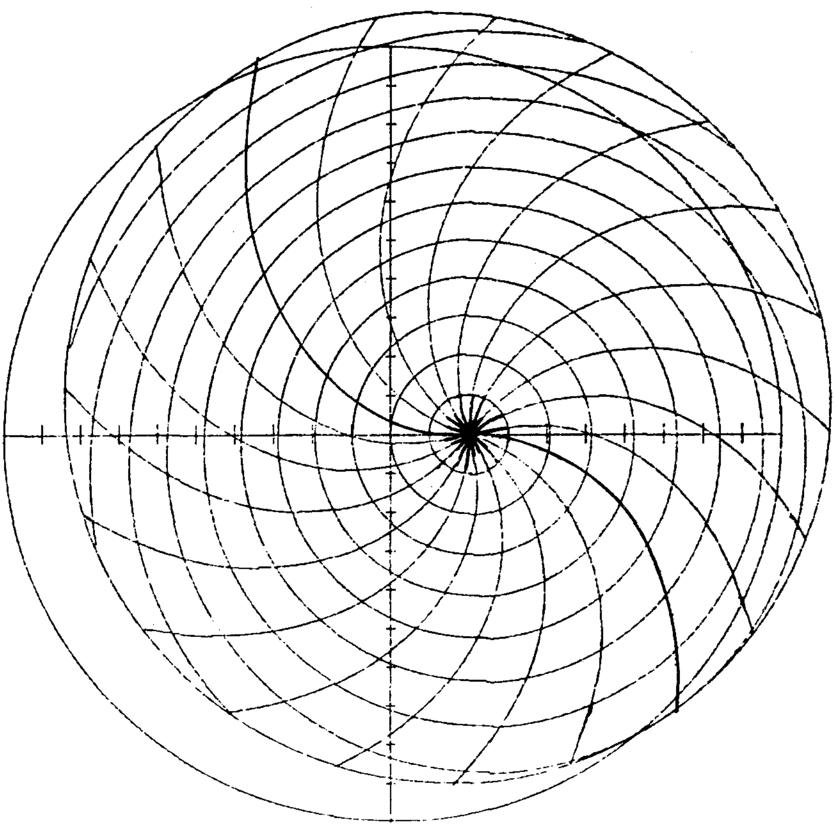


Fig. 4. To illustrate the relationship between optical and geometrical appearances: the optical appearance of a standard grid centered upon (0.2,0) as perceived from that point.

be used to deduce the optical appearance of an object in the disk frame having an outline given in terms of disk polar coordinates, and that neither the center nor the exterior edge of the disk bear any simple relation to their nonrotating analogs.

Finally we may contrast the optical and geometrical appearances of the disk by plotting in Figure 4 with $(\xi, \varphi)=(0.2, 0)$ the optical appearance of a standard grid centered upon the point of observation. Such a graph directly represents the relationship of the two chosen appearances and in particular illustrates the well-known fact that facets of a body geometrically hidden may nonetheless be optically in clear view, due to the recession, in the inertial frame, of the obscuring body from the path of observer-bound photons.

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